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The analysis and synthesis of probability transformers

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SUMMARY
A logical network with n inputs and m outputs ((n, m)-network) supplied by a binary random vector may be considered as a device transforming the input probability distribution into the output one. In this paper a method of analysis of such a network is presented. The approach is based on the orthogonal expansions of the logical functions and the corresponding distributions into the Walsh–Rademacher series. On the basis of the results obtained the synthesis problem has been formulated in terms of the spectral approach.

1. Introduction

The problem of forming a prescribed discrete probability distribution may be formulated in the following way. Given the binary random vector $\boldsymbol{X}=(X_1, X_2, \ldots, X_n)$ with distribution $P(\xi)=P(x_1, x_2, \ldots, x_n)$, $x_i=0, 1$, where ξ is a realization of $\boldsymbol{X}$, it is required to get the binary random vector $\boldsymbol{Y}=(Y_1, Y_2, \ldots, Y_m)$ with distribution $P(\lambda)=P(y_1, y_2, \ldots, y_m)$, $y_i=0, 1$, by a set of Boolean functions $y_i=f_i(\xi)$, $i=1, 2, \ldots, m$, that transform the random variable $\boldsymbol{X}$ into $\boldsymbol{Y}$. This problem though important in digital simulation has only been partially solved for a particular class of distributions. However there is a lack of comprehensive and adequate formal approaches to deal with arbitrary distributions. We show in this paper that the analysis of logical functions in terms of Walsh–Rademacher functions may serve as a tool for solving the synthesis problem.

2. The spectral analysis of the logical network

The application of Walsh–Rademacher functions for the analysis of logical functions was first proposed—though not *expressis verbis*—by Coleman [1]; however it will be useful to recall here the basic definitions and notations.

2.1. *The Walsh–Rademacher functions*

Following Polyac [2] and Fine [3] we define Rademacher functions $R_n(x)$, $x\in[0, 1]$ in the following way

$$R_0 \equiv 1 \;;\; R_n(x) \triangleq \operatorname{sgn}(2^n \pi x) = 1-2x_n \tag{1}$$

where x_n is the n-th position of the binary expansion of x. The Walsh function $W_n(x)$ of the order n_k is defined as

$$W_n(x) \triangleq \prod_{j=1}^{k} R_{n_j}(x) = (1-2x_{n_1})(1-2x_{n_2}) \ldots (1-2x_{n_k}) \tag{2}$$

where

$$n_i < n_{i+1} \quad \text{and} \quad 2n = \sum_{j=1}^{k} 2^{n_j}, \qquad n=1, 2, \ldots \,. \tag{3}$$

The following relations will be important in the sequel

$$W_n(x)\, W_m(x) = W_{n\oplus m}(x)\,, \quad W_n(x)\, W_n(y) = W_n(x\oplus y)\,, \tag{4}$$

where $n \oplus m$ denotes a number obtained as a result of modulo-2 addition of corresponding positions of the binary expansions of number n and m; similarly $z = x \oplus y$ is the result of such operation on numbers x and y. It is clear that every Rademacher function may be rewritten as a Walsh function

$$R_i(x) = W_{2^i-1}(x). \tag{5}$$

Since the set of Walsh functions is orthonormal and closed in L^2 ([2], [3]) then for every function $f(x) \in L^2$ there exists its Walsh–Fourier series

$$f(x) = \sum_{i=0}^{\infty} a_i W_i(x) \tag{6}$$

with the Fourier coefficients a_i

$$a_i = \int_0^1 f(x) W_i(x) dx, \tag{7}$$

which converges to $f(x)$ almost everywhere.

If we consider the partial sum

$$S_N(x) = \sum_{i=0}^{N} a_i W_i(x),$$

then $S_N(x)$ is the best piecewise constant approximation of $f(x)$; it has been shown by Polyac and Schneider [2] that if $N = 2^n - 1$, then $S_{2^n-1}(x)$ is constant on intervals of length $\Delta x_i = 2^{-n}$:

$$S_{2^n-1}(x) = S_{2^n-1}\left(\frac{k}{2^n}\right) \text{ for } x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right), \qquad k = 0, 1, \ldots, 2^n - 1$$

In order to make use of Walsh functions for the analysis of the logical network with n binary inputs $x_1, x_2, \ldots, x_n$ and output y ($(n, 1)$-network) let us notice that there exists a one-to-one correspondence between the set of input vectors

$$\{\xi_j\} \triangleq \{x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)}\}, \qquad j = 0, 1, \ldots, 2^{n-1}$$

and the set of real numbers $\{z_j\} \triangleq \{j \cdot 2^{-n}\}$, $z_j \in [0, 1)$:

$$\{z_j\} \Leftrightarrow \{\xi_j\}. \tag{8}$$

If we thus define the space L_n^2 as the set of functions defined only on the finite set of points $\{z_j\} \triangleq \{j \cdot 2^{-n}\}$ with metric $\rho(u, v)$:

$$\rho(u, v) \triangleq \left(\sum_{j=0}^{2^n-1} [u(z_j) - v(z_j)]^2\right)^{\frac{1}{2}},$$

where $u, v \in L_n^2$, then it is easy to show that the set of Walsh functions is an orthogonal basis in L_n^2 since

$$\sum_{k=0}^{2^n-1} W_i(z_k) W_j(z_k) = 0, \qquad i \neq j$$

$$\sum_{k=0}^{2^n-1} W_i^2(z_k) = 2^n. \tag{9}$$

Because of the equivalence relation (8) every Boolean function $y(\xi) = y(x_1, x_2, \ldots, x_n)$ belongs to the space L_n^2 and therefore may be expanded into the Walsh series

$$y(\xi) = \sum_{i=0}^{2^n-1} a_i W_i(\xi), \qquad \xi \in \{\xi_j\} \tag{10}$$

where

$$a_i = 2^{-n} \sum_{\xi \in \{\xi_j\}} W_i(\xi) y(\xi). \tag{11}$$

Further on the row vector $\boldsymbol{\alpha} \triangleq (a_0, a_1, \ldots, a_{2^n-1})$ will be referred to as the structure of the logical function $y(\xi)$.

2.2. *The probabilistic analysis of logical networks*

Let us assume that the input vector ξ is a realization of the binary random vector $\boldsymbol{X}=(X_1, X_2, \ldots, X_n)$ with probability distribution $P(\xi)$. Thus the (n, m) logical network may be thought of as a device transforming the vector random variable $\boldsymbol{X}$ into the random variable $\boldsymbol{Y}=(Y_1, Y_2, \ldots, Y_m)$ with probability distribution $P(\boldsymbol{\lambda})$ where $\boldsymbol{\lambda} \triangleq (y_1, y_2, \ldots, y_m)$ is a realization of $\boldsymbol{Y}$.

It is clear that $P(\xi)$ and $\boldsymbol{\lambda}(\xi)$ determine the joint distribution $P(\xi, \boldsymbol{\lambda})$ and consequently $P(\boldsymbol{\lambda})$

$$\{P(\xi), \boldsymbol{\lambda}(\xi)\} \Rightarrow P(\xi, \boldsymbol{\lambda}) \Rightarrow P(\boldsymbol{\lambda}) . \tag{12}$$

Since $P(\xi) \in L_n^2$ and $P(\xi, \boldsymbol{\lambda}) \in L_{n+m}^2$ then both distributions can be expanded into a Walsh series,

$$P(\xi) = \sum_{i=0}^{2^n-1} c_i W_i(\xi) \tag{13}$$

$$P(\xi, \boldsymbol{\lambda}) = \sum_{j=0}^{2^{(n+m)}-1} d_j W_j(\xi, \boldsymbol{\lambda}) , \tag{14}$$

where

$$c_i = 2^{-n} \sum_{\{\xi_j\}} P(\xi) W_i(\xi) , \tag{15}$$

$$d_j = 2^{-(n+m)} \sum_{\{\xi_j\} \times \{\lambda_k\}} P(\xi, \boldsymbol{\lambda}) W_j(\xi, \boldsymbol{\lambda}) . \tag{16}$$

Further on the vectors $\boldsymbol{\sigma} \triangleq (c_0, c_1, \ldots, c_{2^n-1})$ and $\boldsymbol{\delta} \triangle (d_0, d_1, \ldots, d_{2^{(n+m)}-1})$ will be referred to as the spectrums of the distributions $P(\xi)$ and $P(\xi, \boldsymbol{\lambda})$ respectively. Now the important question arises whether there exist between these spectrums and the structure of the network relations similar to (12). The answer for the $(n, 1)$-case is established by the following theorem.

Theorem 1. *If $\boldsymbol{\sigma}=(c_0, c_1, \ldots, c_{2^n-1})$ and $\boldsymbol{\delta}^{(1)}=(d_0, d_1, \ldots, d_{2^{(n+1)}-1})$ are the spectrums of the distributions $P(\xi)$ and $P(\xi, y)$ respectively, then*

$$\boldsymbol{\delta}_1^{(1)} = \tfrac{1}{2}\boldsymbol{\sigma} , \tag{17}$$

$$\boldsymbol{\delta}_2^{(1)} = \boldsymbol{\delta}_1^{(1)} - \boldsymbol{\alpha} C_{\oplus} , \tag{18}$$

where $\boldsymbol{\delta}_1^{(1)}$ and $\boldsymbol{\delta}_2^{(1)}$ are 2^n-elements segments of the vector $\boldsymbol{\delta}^{(1)}$, which in turn is the concatenation of $\boldsymbol{\delta}_1^{(1)}$ and $\boldsymbol{\delta}_2^{(1)}$:

$$\boldsymbol{\delta}^{(1)} \triangleq (\boldsymbol{\delta}_1^{(1)}, \boldsymbol{\delta}_2^{(1)}) ,$$

$$\boldsymbol{\delta}_1^{(1)} \triangleq (d_0, d_1, \ldots, d_{2^n-1}) , \quad \boldsymbol{\delta}_2^{(1)} \triangleq (d_{2^n}, d_{2^m+1}, \ldots, d_{2^{(n+1)}-1})$$

and $C_{\oplus}$ denotes the matrix with entries c_{ij} being the result of the modulo-2 permutation of elements of the spectrum $\boldsymbol{\sigma}$:

$$c_{ij} = c_{i \oplus j} .$$

The interpretation of the $i \oplus j$ operation is the same as in eq. (4).

Proof. First of all let us notice that

$$W_k(\xi, y) = W_k(\xi) \quad \text{for} \quad k = 0, 1, \ldots, 2^n-1 ,$$

$$W_k(\xi, y) = W_l(\xi) \cdot W_{2^n}(\xi, y) = W_l(\xi) \cdot R_{n+1}(y)$$

$$\text{for} \quad k = 2^n+l, \; l = 0, 1, \ldots, 2^n-1 .$$

Taking this into account we have from (16) for $i=0, 1, \ldots, 2^n-1$

$$d_i = 2^{-(n+1)} \sum_{\{\xi_j\}} W_i(\xi) [P(\xi, 0) + P(\xi, 1)] = 2^{-(n+1)} \sum_{\{\xi_j\}} P(\xi) W_i(\xi) ,$$

which when compared with (14) completes the proof of the first part of Theorem 2, (eq. 17), $d_i = \frac{1}{2} c_i$, $i=0, 1, \ldots, 2^n-1$. For higher order elements d_k of the spectrum $\boldsymbol{\delta}^{(1)}$ let us write $k=2^n+l$, $l=0, 1, \ldots, 2^n-1$; therefore (16) will take the form

$$d_k = 2^{-(n+1)} \sum_{\{\xi_j\}} [P(\xi, 0) - P(\xi, 1)]\, W_k(\xi). \tag{19}$$

If we notice that for any logical function $y(\xi)$

$$\Pr(Y=1\,|\,\xi) = y(\xi)\,, \quad \Pr(Y=0\,|\,\xi) = 1 - y(\xi)\,, \tag{20}$$

then taking into account that $P(\xi, y) = P(\xi)\, P(y/\xi)$, we may substitute into (19) the relations (20) replacing simultaneously $y(\xi)$ and $P(\xi)$ by their orthogonal expansions. Having performed the necessary transformations we obtain

$$d_k = 2^{-(n+1)} \sum_{\xi \in \{\xi_j\}} W_l(\xi) \sum_{i=0}^{2^n-1} c_i W_i(\xi) \left[1 - 2 \sum_{j=0}^{2^n-1} a_j W_j(\xi)\right].$$

If we note that from definition of the Walsh functions

$$\sum_{\{\xi_j\}} W_k(\xi) = \begin{cases} 0\,, & \text{if} \quad k = 1, 2, \ldots, 2^n - 1 \\ 2^n & \text{if} \quad k = 0 \end{cases} \tag{21}$$

and

$$W_l(\xi)\, W_i(\xi) = W_{l \oplus i}(\xi)\,,$$

then we get at last

$$d_k = d_{2^n + l} = \tfrac{1}{2} c_l - \sum_{i=0}^{2^n-1} a_i c_{i \oplus l}\,, \qquad l = 0, 1, \ldots, 2^n - 1$$

or, on the basis of (17)

$$d_{2^n + l} = d_l - 2 \sum_{i=0}^{2^n-1} a_i a_{i \oplus l}\,, \qquad i, l = 0, 1, \ldots, 2^n - 1\,.$$

This result constitutes the second part (18) of Theorem 2.

In order to extend the results of Theorem 1 to the (n, m)-case with m output functions $y_1, y_2, \ldots, y_m$ let us notice that $\boldsymbol{\alpha} C_{\oplus}$ denotes the binary operation on two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$ with the result being a vector too. For example, if $\boldsymbol{\alpha} = (a_0, a_1, a_2, a_3)$, $\boldsymbol{\sigma} = (c_0, c_1, c_2, c_3)$, then

$$\begin{aligned} \boldsymbol{\alpha} C_{\oplus} = (&a_0 c_0 + a_1 c_1 + a_2 c_2 + a_3 c_3\,,\ a_0 c_1 + a_1 c_0 + a_2 c_3 + a_3 c_2\,, \\ &a_0 c_2 + a_1 c_3 + a_2 c_0 + a_3 c_1\,,\ a_0 c_3 + a_1 c_2 + a_2 c_1 + a_3 c_0)\,. \end{aligned}$$

We shall denote this operation as $\otimes$:

$$\boldsymbol{\alpha} \otimes \boldsymbol{\sigma} \triangleq \boldsymbol{\alpha} C_{\oplus}\,. \tag{22}$$

It is easy to verify that this operation is commutative and associative:

$$\boldsymbol{\alpha} \oplus \boldsymbol{\sigma} = \boldsymbol{\sigma} \oplus \boldsymbol{\alpha} = \boldsymbol{\sigma} A_{\oplus} = \boldsymbol{\alpha} C_{\oplus}\,, \qquad \boldsymbol{\alpha} \oplus (\boldsymbol{\beta} \oplus \boldsymbol{\gamma}) = (\boldsymbol{\alpha} \oplus \boldsymbol{\beta}) \oplus \boldsymbol{\gamma}\,. \tag{23}$$

Theorem 2. *The spectrum $\boldsymbol{\delta}^{(m)}$ of the joint distribution $P(\xi, \boldsymbol{\lambda})$, $\xi = (x_1, x_2, \ldots, x_m)$, $\boldsymbol{\lambda} = (y_1, y_2, \ldots, y_m)$ consists of 2^m segments ρ_i, $i = 0, 1, \ldots, 2^m - 1$ each containing 2^n elements. Every segment corresponds to the subset of the output variables $\{y_j\}$, $j = 1, 2, \ldots, m$ and can be computed by the following formula*

$$\rho_i = \sum_{l=1}^{r_i} \sum_{j=1}^{\binom{r_i}{l}} \frac{(-1)^l}{2^{m-l}}\, \boldsymbol{\sigma} \otimes \boldsymbol{\alpha}^{(i_1(j))} \times \boldsymbol{\alpha}^{(i_2(j))} \otimes \ldots \otimes \boldsymbol{\alpha}^{(i_l(j))} + \frac{\boldsymbol{\sigma}}{2^m}\,, \tag{24}$$

where

r_i is the number of outputs y_{i_j}, $j = 1, 2, \ldots, r_i$, $r_i = 1, 2, \ldots, m$ associated with the segment ρ_i, referred to as the rank of ρ_i;
$\boldsymbol{\alpha}^{(i_k(j))}$ is the structure of the function $y_{i_k}(\xi)$ for a given combination j of output variables;
i_k are the positions of the binary expansion of i that are equal to 1, i.e.

$$i = \sum_{k=1}^{r_i} i_k \cdot 2^k$$

Proof. Let us rewrite the logical function $y_k(\xi)$ in the following form

$$y_k(\xi) = f[\xi, y_1, y_2, \dots, y_m]$$

and consider the structure $\boldsymbol{\beta}^{(k)} = (b_0, b_1, \dots, b_{2^{(n+k)}-1})$. If we denote by $\boldsymbol{\delta}^{(k)}$ the spectrum of the joint distribution $P(\xi, y_1, y_2, \dots, y_k)$ then on the basis of (17) and (18) we can write the recurrent equations

$$\boldsymbol{\delta}_1^{(k)} = \tfrac{1}{2}\boldsymbol{\delta}^{(k-1)}\ ; \quad \boldsymbol{\delta}^{(0)} \equiv \boldsymbol{\sigma}$$

$$\boldsymbol{\delta}_2^{(k)} = \boldsymbol{\delta}_1^{(k)} - 2\boldsymbol{\delta}_1^{(k)} \quad B_{\oplus}^{(k)} = \tfrac{1}{2}\boldsymbol{\delta}^{(k-1)} - \boldsymbol{\delta}^{(k-1)} B_{\oplus}^{(k)}$$

where $\boldsymbol{\delta}^{(k)} \triangleq (\boldsymbol{\delta}_1^{(k)}, \boldsymbol{\delta}_2^{(k)})$ is the concatenation of the two $2^{(n+k-2)}$-elements vectors $\boldsymbol{\delta}_1^{(k)}, \boldsymbol{\delta}_2^{(k)}$ and the matrix $B_{\oplus} = [b_{ij}] \triangleq [b_{i \oplus j}]$. However, because the function $y_k(\xi)$ is in fact independent of the variables y_i, the structure has the following form

$$\boldsymbol{\beta}^{(k)} = (\boldsymbol{\alpha}^{(k)}, O_1, O_2, \dots, O_{2^k-1})$$

where $O_i = (0, 0, \dots, 0)$ are 2^n-zero-elements vectors. Therefore the matrix $B_{\oplus}^{(k)}$ is a quasi-diagonal one, namely

$$B_{\oplus}^{(k)} = \left.\begin{bmatrix} A_{\oplus}^{(k)} & & & \\ & A_{\oplus}^{(k)} & & \\ & & \ddots & \\ & & & A_{\oplus}^{(k)} \end{bmatrix}\right\} 2^{(k-1)} \text{ matrix-rows}$$

It is thus easy to verify that the number of segments of the spectrum $\boldsymbol{\delta}^{(k)}$ is equal to 2^k:

$$\boldsymbol{\delta}^{(k)} = (\rho_0, \rho_1, \dots, \rho_{2^k-1})\,, \quad \rho_i = (d_{i \cdot 2^n}\,, \ d_{i \cdot 2^n+1}\,, \dots, \ d_{(i+1)2^n-1})$$

For $k=0$, we have from (25)

$$\boldsymbol{\delta}^{(0)} = \rho_0 = \boldsymbol{\sigma}\,, \quad \boldsymbol{\delta}_2^{(1)} = \tfrac{1}{2}\boldsymbol{\sigma} - \boldsymbol{\sigma} \otimes \boldsymbol{\alpha}^{(1)}\,, \quad \boldsymbol{\delta}^{(1)} = (\tfrac{1}{2}\boldsymbol{\sigma}\,, \ \tfrac{1}{2}\boldsymbol{\sigma} - \boldsymbol{\sigma} \otimes \boldsymbol{\alpha}^{(1)})\,,$$

$$\boldsymbol{\delta}^{(2)} = (\tfrac{1}{4}\boldsymbol{\sigma}\,, \ \tfrac{1}{4}\boldsymbol{\sigma} - \tfrac{1}{2}\boldsymbol{\sigma} \otimes \boldsymbol{\alpha}^{(1)}\,,$$

$$\tfrac{1}{4}\boldsymbol{\sigma} - \tfrac{1}{2}\boldsymbol{\sigma} \otimes \boldsymbol{\alpha}^{(2)}\,, \ \tfrac{1}{4}\boldsymbol{\sigma} - \tfrac{1}{2}\boldsymbol{\sigma} \otimes \boldsymbol{\alpha}^{(1)} - \tfrac{1}{2}\boldsymbol{\sigma} \otimes \boldsymbol{\alpha}^{(2)} + \boldsymbol{\sigma} \otimes \boldsymbol{\alpha}^{(1)} \otimes \boldsymbol{\alpha}^{(2)})\,.$$

Repeating this procedure recurrently for $k=3, 4, \dots, m$ we get expression (24) for ρ_i.

It is clear that the elements g_i of the spectrum $\boldsymbol{g} = (g_0, g_1, \dots, g_{2^m-1})$ of the distribution $P(\boldsymbol{\lambda}) = P(y_1, y_2, \dots, y_m)$ are proportional to the first elements of the segments ρ_i namely

$$g_i = 2^n \cdot d_{i \cdot 2^n}\,, \qquad i = 0, 1, \dots, 2^m - 1$$

$$g_i = \varphi_i(\boldsymbol{\alpha}^{(1)}\,, \ \boldsymbol{\alpha}^{(2)} \dots, \boldsymbol{\alpha}^{(m)})\,. \tag{26}$$

The problem of existence of any network which transforms the input white variable $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ with uniform distribution $P(\xi) = 2^{-n}$ and position-independency into the variable $\boldsymbol{Y}$ has been solved by Warfield [4], [5]. He introduced the concept of the stochastic degree of the random vector $\boldsymbol{Y}$ as the least integer $S(\boldsymbol{Y})$ such that for every $\boldsymbol{\lambda}_j$

$$P(\boldsymbol{\lambda}_j) = k_j \cdot 2^{-S(\boldsymbol{Y})}\,,$$

where k_i, $i = 0, 1, \dots, 2^m - 1$ are integers satisfying the condition

$$\sum_{i=0}^{2^m-1} k_i = 2^{S(\boldsymbol{Y})}\,.$$

In other words the stochastic degree may be interpreted as the length of the binary representation of $P(\boldsymbol{\lambda}_j)$. Warfield has showed that if the stochastic degree $S(\boldsymbol{Y})$ of the output vector $\boldsymbol{Y}$ is finite and less then the stochastic degree $S(\boldsymbol{X})$ of the input vector $\boldsymbol{X}$ then there exist a logical network (n, m) where $n = S(\boldsymbol{X})$, $m = S(\boldsymbol{Y})$ which transforms $P(\xi)$ into $P(\boldsymbol{\lambda})$.

Further on distributions with finite stochastic degree will be referred to as *binary realizable* ones since only such distributions may be obtained by a logical transformation of the white source $\boldsymbol{X}$ with $P(\xi) = 2^{-n}$.

Therefore for $n > m$ the synthesis problem may be formulated in terms of the spectral analysis as solutions of the equations (26) with respect to the set of structures $\omega \triangle \langle \boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \ldots, \boldsymbol{\alpha}^{(m)} \rangle$ for given values of the spectrum $\gamma = (g_0, g_1, g_2, \ldots, g_{2^m-1})$ of the binary realizable distribution $P(\boldsymbol{\lambda})$. However, equations (26) may be in general satisfied by ω not being the set of structures of logical functions.

Consequently the question arises, what kind of conditions should be satisfied by an arbitrary 2^n-elements vector $\boldsymbol{\beta} = (b_0, b_1, \ldots, b_{2^n-1})$ to represent a logical function. These conditions are established by the following theorem.

Theorem 3. *A necessary and sufficient condition for any 2^n-elements vector $\boldsymbol{\chi} = (k_0, k_1, \ldots, k_{2^n-1})$ to represent a logical function is to satisfy the set of equations*

$$k_j = \sum_{i=0}^{2^n-1} k_i k_{i\oplus j}, \qquad j = 0, 1, \ldots, 2^n - 1 \tag{27}$$

Proof. First of all, let us note that a necessary and sufficient condition for any function $y(\xi)$ to be a logical one i.e. to take values 0 or 1 only is to satisfy the following equation

$$y(\xi) = y^2(\xi) \tag{28}$$

for every $\xi \in \{\xi_j\}$.

In order to prove the necessity of the conditions (27) for a given logical function $y(\xi)$ let us substitute its Walsh series into the expression $z(\xi) = y^2(\xi)$ and take into account relation (4):

$$z(\xi) = \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^n-1} k_i k_j W_{i\oplus j}(\xi) \,. \tag{29}$$

Assuming that

$$z(\xi) = \sum_{k=0}^{2^n-1} b_k W_k(\xi) \,,$$

we have

$$b_k = 2^{-n} \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^n-1} k_i k_j \sum_{\{\xi_j\}} W_k(\xi) \,,$$

or on the basis of (21)

$$b_j = \sum_{i=0}^{2^n-1} k_i k_{i\oplus j} \,.$$

To prove the sufficiency of these conditions let us assume that there exists a function $z(\xi)$, with spectrum $\boldsymbol{\beta} = (b_0, b_1, \ldots, b_{2^n-1})$ satisfying eqs. (27), which is not a logical one i.e. there is at least one vector $\xi_k \in \{\xi_j\}$ for which $z(\xi_k) \neq z^2(\xi_k)$. Squaring the sum $z(\xi) = \sum_i b_i W_i(\xi)$ and finding the structure of $z^2(\xi)$ we get (29); making then use of (27) we obtain (28).

It is easy to see that the conditions (27) may be rewritten in the matrix form

$$\boldsymbol{\alpha} A_{\oplus} = \boldsymbol{\alpha} \tag{30}$$

$$\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} = \boldsymbol{\alpha} \tag{31}$$

The results of this section may be summarized in the form of

Theorem 4. *A necessary and sufficient condition for any set of structures $\omega = \langle \boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \ldots, \boldsymbol{\alpha}^{(m)} \rangle$ to represent the transformation of the white vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ into the binary realizable vector $Y = (Y_1, Y_2, \ldots, Y_m)$ with distribution $P(\boldsymbol{\lambda}) = P(y_1, y_2, \ldots, y_m)$ and stochastic degree $m < n$ is to satisfy the following set of equations*

$$\begin{array}{ll} \boldsymbol{\alpha}^{(1)} \otimes \boldsymbol{\alpha}^{(1)} = \boldsymbol{\alpha}^{(1)}, & g_0 = \varphi_0(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \ldots, \boldsymbol{\alpha}^{(m)}) \\ \boldsymbol{\alpha}^{(2)} \otimes \boldsymbol{\alpha}^{(2)} = \boldsymbol{\alpha}^{(2)}, & g_1 = \varphi_1(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \ldots, \boldsymbol{\alpha}^{(m)}) \\ \cdots\cdots & \cdots\cdots \\ \boldsymbol{\alpha}^{(m)} \otimes \boldsymbol{\alpha}^{(m)} = \boldsymbol{\alpha}^{(m)}, & g_{2^m-1} = \varphi_{2^m-1}(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \ldots, \boldsymbol{\alpha}^{(m)}) \end{array} \tag{32}$$

where $g_0, g_1, \ldots, g_{2^m-1}$ *is the spectrum of* $P(\lambda)$ *and the functions* $\varphi_i(\omega)$, $i=0, 1, \ldots, 2^m-1$ *are defined by* (26).

4. Conclusions

In the previous sections we have formulated the analysis and synthesis problem in terms of the spectral approach. However the results are not completed—particularly there is a need of a methodical algorithm for solving the set of equations (32). It would by very useful to reformulate this algorithm into the form of logical equations and implement it in the form of a computer program. There is a perspective of doing this based upon the following circumstances.

Let us note that the system $\Omega \triangleq \langle S_n, \otimes, \boldsymbol{1}\rangle$ where S_n is the set of all 2^n-elements vectors is a semigroup with unity $\boldsymbol{1}=(1, 0, 0, \ldots, 0)$. If we consider the system $\phi=\langle f_n, \cdot, 1_f\rangle$ where f_n is the set of all functions defined on the set $\{\xi_j\}$, $j=0, 1, \ldots, 2^n-1$; "$\cdot$" is the operation of multiplication in f_n and 1_f is the function identically equal to 1, then the transformation $T: \Omega \to \phi$ is an isomorphism Ω onto ϕ. Moreover if we consider the subset $\alpha_n \subset S_n$, such that for every $\alpha \in \alpha_n$, $\alpha \otimes \alpha = \alpha$ holds then the transformation T transforms α_n into the subset $l_n \subset f_n$ of all logical functions of n variables. Therefore all expressions in terms of the spectral analysis have their equivalents in the Boolean equations domain.

Although the applications of Theorem 4 are only restricted to a white-source at the input of the network and to the binary-realizable distributions at the output they can be extended to include these cases too. However the solution will be then the best least-square approximation of the required one.

Let us also note that because of the commutativity of the operation $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ it is possible to formulate the reverse synthesis problem, namely: given a prescribed distribution $P(\boldsymbol{\lambda})$ and a (n, m)-network to find an input distribution $P(\xi)$ that yields $P(\boldsymbol{\lambda})$.

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